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TOPICAL REVIEW

Symmetries of the turbulent state

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Abstract
The emphasis of this review is on fundamental properties, degree of universality and symmetries of the turbulent state. The central questions are which symmetries remain broken even when the symmetry-breaking factor reaches zero, and which symmetries, in contrast, emerge in the state of developed turbulence. We shall see that time reversibility is broken in all cases since turbulence is a far-from-equilibrium state accompanied by dissipation. As far as scale invariance is concerned, we argue that it is always broken in direct cascades (toward small scales) no matter how far one goes away from the pumping scale. In contrast, inverse cascades become scale invariant as they go toward large scales. Moreover, some properties of the inverse cascades seem to be conformal invariant and related to Schramm–Loewner evolution (a class of random curves that can be mapped to a 1D Brownian walk).

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

We define turbulence as a state of a physical system with many interacting degrees of freedom deviated far from equilibrium. This state is irregular both in time and in space and is accompanied by dissipation. Developed turbulence corresponds to the case when the scales of externally excited and effectively dissipated motions are vastly different. For example, a moving car leaves behind meter-size vortices while viscous friction is only effective for eddies smaller than a fraction of a millimeter. Instabilities of large vortices, their breakdown and fragmentation bring energy from input to dissipation scales by a cascade. Cascade must be a natural state of any nonlinear system where input and output are far away as long as the interaction is effectively local. Locality here means that effective energy exchange between different modes reaches zero with the ratio of their scales. Apart from energy, other quantities conserved by interaction can cascade too. For example, during ore pulverization (when colliding stones are broken) mass cascades toward smaller sizes, while in water droplet coagulation (say, in clouds) mass cascades toward larger sizes. The cascade toward small scales is usually called direct while that toward large scales is called inverse. If a system has more than one conservation law in the absence of input and dissipation, then input at some scale can generate both direct and inverse cascades simultaneously, as happens for two-dimensional vortex turbulence according to Kraichnan [36] or wave turbulence on a water surface according to Zakharov [13, 58]. The interval of scales between input and output is called the inertial interval (or transparency window).

Developed turbulence contains many excited degrees of freedom and requires a statistical description. In most cases, many strongly interacting degrees of freedom can be neither described theoretically nor satisfactory modeled on computer. Therefore, general symmetry aspects of turbulence statistics are of prime importance. Relevant symmetries are that of the measure. Consider, for instance, the probability distribution $\mathcal{P}(\delta v, \mathbf{r})$ of the longitudinal velocity difference $\delta v = (\delta \mathbf{v} \cdot \mathbf{r})/r$ measured at two points separated by the distance $\mathbf{r}$. One asks whether the distribution is

- time reversible, $\mathcal{P}(\delta v, \mathbf{r}) = \mathcal{P}(-\delta v, \mathbf{r})$,
- isotropic, $\mathcal{P}(\delta v, \mathbf{r}) = \mathcal{P}(\delta v, r)$,
- scale invariant, $\mathcal{P}(\delta v, r) = (\delta v)^{-1} f(\delta v/r^h)$.

One can also ask if the statistics is invariant with respect to Galilean transform $\mathbf{v} \to \mathbf{v} + \text{const}$. To keep a system in a turbulent state (i.e. away from equilibrium) one needs some external force, which generally breaks isotropy and scale invariance. Turbulence in a statistically steady state is necessarily accompanied by dissipation which breaks time reversibility. Yet these two mechanisms act on vastly different scales so one asks if the symmetries are restored in the inertial interval (so that some information on pumping and dissipation is forgotten). One can use the cascade idea to establish non-restoration of time reversibility. For an incompressible fluid, the energy flux (per unit mass) ϵ through the given scale r can be estimated via the velocity difference δv measured at that scale as the energy $(\delta v)^2$ divided by the time $r/\delta v$. Requirement of the flux constancy across the scales gives $(\delta v)^3 \sim \epsilon r$. Of course, δv is a fluctuating quantity and we ought to make statements on its moments or probability distribution $\mathcal{P}(\delta v, r)$. Energy flux constancy fixes the third moment, $\langle (\delta v)^3 \rangle \sim \epsilon r$. Since velocity changes sign under time reflection then a nonzero third moment means that time reversibility is broken. Assuming that the energy input rate ϵ is independent of viscosity we see that the energy flux constancy is an example of an anomaly: the effect of symmetry breaking remains finite when the symmetry-breaking factor (viscosity) reaches zero. This so-called dissipative anomaly has been discovered by Kolmogorov in 1941 so it is probably the first anomaly in physics; conceptually it is similar to the axial anomaly in quantum field theory as first noted by Polyakov [47]. As we have shown, the cascade idea can indeed be turned into an exact relation which requires the flux (the third-order moment) to be constant across the inertial interval of scales.

One can now follow Kolmogorov (Onsager, Heizenberg and many others) and try to use the flux relation to guess the scaling properties of turbulence, in particular, the index h assuming the probability distribution is scale invariant. Is it enough to know just the flux, i.e. the input rate of energy (or other quantity) in a statistical steady state? A positive answer would mean universality of turbulence, i.e. independence of the details of the pumping. The answer on scale invariance and universality is 'definitely no for direct cascades' and 'probably yes for inverse cascades', as discussed below.

2. Direct cascades

Here we consider direct cascades and discover that the symmetries broken by pumping and dissipation are not restored in the inertial interval.

2.1. Burgers and KPZ

Consider arguably the simplest hydrodynamic system. In the reference frame moving with the sound velocity c, the velocity u of a weakly compressible 1D flow ($u \ll c$) satisfies the Burgers equation [23, 29, 39]:

$$u_t + uu_x - \nu u_{xx} = 0. \tag{1}$$

This equation can be also written for the potential h defined by $u = \nabla h$, then it can be considered in multi-dimensional versions as well when it describes surface growth, directed polymer etc. Such a form

$$h_t + (\nabla h)^2/2 - \nu\Delta h = \xi \tag{2}$$

is called the Kardar–Parisi–Zhang (KPZ) equation when the driving force is white both in time and in space: $\langle\xi(\mathbf{x}, t)\xi(0, 0)\rangle = T\delta(t)\delta(\mathbf{x})$. To have a benchmark for comparison, let us describe briefly the properties of such a state driven at small scales before we turn to turbulence. This state has a simple Gaussian equilibrium (Gibbs) single-point probability distribution: $\mathcal{P}(h) \sim \exp(-U/T)$ where the energy $U = \int h_x^2\,\mathrm{d}x\big/2$ is conserved by the nonlinear term in 1D. Indeed, the Fokker–Planck equation for the probability density function (PDF)

$$\frac{\partial\mathcal{P}(h,t)}{\partial t} = \int \mathrm{d}x \frac{\delta}{\delta h(x)}\left[T\frac{\delta\mathcal{P}}{\delta h(x)} - \mathcal{P}\left(\frac{\delta U}{\delta h(x)} + \frac{h_x^2}{2}\right)\right] \tag{3}$$

has such a solution since $\int \mathrm{d}x\, h_x^2\delta U\big/\delta h = \int \mathrm{d}x\, h_x^2 h_{xx} = 0$ and $\int \mathrm{d}x\, \delta h_x^2\big/\delta h(x) = 0$. Gaussian statistics is completely determined by the second moment which behaves in a diffusive way

$$\langle[h(x) - h(0)]^2\rangle = \int_{-\infty}^{\infty}[\exp(\mathrm{i}qx) - 1]\frac{T}{q^2}\frac{\mathrm{d}q}{2\pi} = T|x|. \tag{4}$$

The probability density function (PDF) for the velocities is thus steady, time reversible and scale invariant.

Consider now the Burgers turbulence driven by a large-scale force or appearing from a large-scale initial distribution. Note that (1) has a propagating shock-wave solution $u = v\tanh[v(x - vt)/2\nu]$ with the energy dissipation rate $\nu\int u_x^2\,\mathrm{d}x$ independent of ν. The shock width ν/v is a dissipative scale and we consider acoustic turbulence produced by pumping correlated on much larger scales (for example, pumping a pipe from one end by frequencies much less than cv/ν) so that the Reynolds number is large. After some time, it will develop shocks at random positions. Structure functions, $S_n(x, t) = \langle[u(x, t) - u(0, t)]^n\rangle$, can be readily determined assuming that Burgers turbulence consists of shocks separated by smooth parts. In this case, $S_n(x) \sim C_n|x|^n + C_n'|x|$ where the first term comes from the regular (smooth) parts of the velocity while the second comes from $O(x)$ probability to have a shock in the interval x. The scaling exponents, $\xi_n = \mathrm{d}\ln S_n/\mathrm{d}\ln x$, thus behave as follows: $\xi_n = n$ for $n \leqslant 1$ and $\xi_n = 1$ for $n > 1$. Moreover, we can find not only scaling exponents but also factors since all the moments with $n > 1$ are determined solely by shocks at the limit $x \to 0$. Take $u = v\tanh(vx/2\nu)$ and get $S_3 = 8v^3x/L$ and $\epsilon_2 = \nu\langle u_x^2\rangle = L^{-1}\nu\int_{-L/2}^{L/2}u_x^2\,\mathrm{d}x = 2v^3/3L$ which gives the energy flux relation

$$S_3 = -12\epsilon_2 x. \tag{5}$$

We see that indeed the third moment is expressed solely via the mean energy dissipation rate which is the energy flux. Further, denote $\epsilon_4 = 6\nu\big[\langle u^2u_x^2\rangle + \langle u^2\rangle\langle u_x^2\rangle\big]$ and get $\langle u^2u_x^2\rangle = 2v^5/15L$ so that $\epsilon_4 = -24v^5/5L$. Substituting $v^5/L = 5\epsilon_4/24$ into $S_5 = 32v^5x/L$ we get

$$S_5 = -20\epsilon_4 x/3 = -40\nu x\big[\langle u^2u_x^2\rangle + \langle u^2\rangle\langle u_x^2\rangle\big]. \tag{6}$$

One can derive such relations from the equation on the structure functions

$$\frac{\partial S_{2n}}{\partial t} = -\frac{2n-1}{2n+1}\frac{\partial S_{2n+1}}{\partial x} - 4\epsilon_n + \nu\frac{\partial^2 S_{2n}}{\partial x^2}. \tag{7}$$

Here we denoted $\langle \dot{E}_n\rangle = \epsilon_n$ the viscous dissipation rates of the integrals $E_n = \int u^{2n}\,\mathrm{d}x/2$ which are conserved by the inviscid Burgers equation (see, e.g., [48]). For example, consider an unforced case and $n = 2$, then write $\partial_t S_4/4 = -(3/20)\partial_x S_5 - 6\nu\big[\langle u^2u_x^2\rangle + \langle u_1^2u_{2x}^2\rangle\big] + 12\nu\langle u_1u_2u_{1x}^2\rangle + 2\nu\langle u_1^3u_{2xx}\rangle$. Since the distance x_{12} is in the inertial interval then we can neglect $\langle u_1^3u_{2xx}\rangle$ and $\langle u_1u_2u_{1x}^2\rangle$, and we can put $\langle u_1^2u_{2x}^2\rangle \approx \langle u^2\rangle\langle u_x^2\rangle$.

Equation (7) describes both a free decay (then ϵ_n depends on t) and the case of a permanently acting pumping which generates turbulence statistically steady at scales less than the pumping length. In both cases we can neglect the left-hand side (in the first case, $\partial S_{2n}/\partial t \simeq S_{2n}u/L \ll \epsilon_n \simeq u^{2n+1}/L$ where L is a typical distance between shocks) while in the second case $\partial S_2/\partial t = 0$. Consider now limit $\nu \to 0$ at fixed x (and t for decaying turbulence). Shock dissipation provides for a finite limit of ϵ_n at $\nu \to 0$ then

$$S_{2n+1} = -4\frac{2n+1}{2n-1}\epsilon_n x. \tag{8}$$

This equation shows that all possible symmetries are broken. First, nonzero odd moments of the velocity mean time irreversibility. Second, neither E_n, ϵ_n nor S_{2n+1} are Galilean invariant for $n > 2$, see also [10]. Third, the PDF is not scale invariant, that is the function of the re-scaled velocity difference $\delta u/x^h$ cannot be made scale independent for any h. Breakdown of scale invariance means that the low-order moments decrease faster than the high-order ones as one goes to smaller scales, i.e. the smaller the scale the more probable are large fluctuations and the statistics is getting more and more non-Gaussian. In other words, the probability of strong fluctuations increases with the resolution. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when $x/L \to 0$.

We thus conclude that the PDF of the velocity differences in the inertial interval depends on the infinitely many pumping related parameters, the fluxes of all dynamical integrals of motion. Simple bi-modal nature of the Burgers turbulence (shocks and smooth parts) means that the PDF is actually determined by two (non-universal) functions, each depending on a single argument: $P(\delta u, x) = \delta u^{-1} f_1(\delta u/x) + x f_2(\delta u/u_{\mathrm{rms}})$. Note that $S_2(x) \propto |x|$ corresponds to $E(k) \propto k^{-2}$, since every shock gives $u_k \propto 1/k$ at $k \ll v/\nu$, that is the energy spectrum is determined by the type of structures (shocks) rather than by energy flux constancy. That is, Burgers turbulence demonstrates universality of a different kind: the type of structures that dominate turbulence (here, shocks) is universal while the statistics of their amplitudes depends on pumping.

2.2. *3D Navier–Stokes turbulence*

An incompressible fluid flow is described by the Navier–Stokes equation

$$\partial_t \mathbf{v}(\mathbf{r},t) + \mathbf{v}(\mathbf{r},t)\cdot\nabla\mathbf{v}(\mathbf{r},t) - \nu\nabla^2\mathbf{v}(\mathbf{r},t) = -\nabla p(\mathbf{r},t), \quad \operatorname{div}\mathbf{v} = 0. \tag{9}$$

We are again interested in the structure functions $S_n(\mathbf{r},t) = \langle[(\mathbf{v}(\mathbf{r},t) - \mathbf{v}(0,t))\cdot\mathbf{r}/r]^n\rangle$ and consider distance r smaller than the force correlation scale for a steady case and smaller

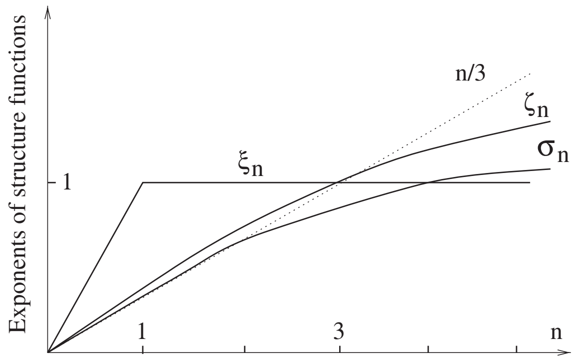

Figure 1. The scaling exponents of the structure functions ξ_n for Burgers, ζ_n for 3D Navier–Stokes and σ_n for the passive scalar. The dotted straight line is $n/3$.

than the size of the turbulent region for a decay case. Similar to (7), one can derive the Karman–Howarth relation between S_2 and S_3 (see [39]):

$$\frac{\partial S_2}{\partial t} = -\frac{1}{3r^4}\frac{\partial}{\partial r}(r^4 S_3) + \frac{4\epsilon}{3} + \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right). \tag{10}$$

Here $\epsilon = \nu\langle(\nabla \mathbf{v})^2\rangle$ is the mean energy dissipation rate. Neglecting the time derivative (which is zero in a steady state and small compared to ϵ for decaying turbulence) one can multiply (10) by r^4 and integrate: $S_3(r) = -4\epsilon r/5 + 6\nu\, \mathrm{d}S_2(r)/\mathrm{d}r$. Kolmogorov considered the limit $\nu \to 0$ for fixed r and *assumed* nonzero limit for ϵ which gives the so-called 4/5 law [28, 35, 39]:

$$S_3 = -\tfrac{4}{5}\epsilon r. \tag{11}$$

Similar to (5), this relation means that the kinetic energy has a constant flux in the inertial interval of scales (the viscous scale η is defined by $\nu S_2(\eta) \simeq \epsilon\eta^2$). Let us stress that this flux relation is built upon the assumption that the energy dissipation rate ϵ has a nonzero limit at vanishing viscosity. Since the input rate can be independent of viscosity, this is the assumption needed for an existence of a steady state at the limit: no matter how small the viscosity, or how high the Reynolds number or how extensive the scale-range participating in the energy cascade, the energy flux is expected to remain equal to that injected at the stirring scale. Unlike compressible (Burgers) turbulence, here we do not know the form of the specific singular structures that are supposed to provide non-vanishing dissipation in the inviscid limit (as shock waves do). Experimental data show, however, that the dissipation rate is indeed independent of the Reynolds number when $Re \gg 1$ which means again dissipative anomaly and time irreversibility of the velocity statistics even at scales far exceeding the viscous scale. If one screens a movie of steady turbulence backward, it looks wrong.

The law (11) shows that the third-order moment is universal, i.e. it does not depend on the details of the turbulence production but is determined solely by the mean energy dissipation rate. The rest of the structure functions have never been derived. Experiments suggest that $\zeta_n = \mathrm{d}\ln S_n/\mathrm{d}\ln r$ lie on a smooth concave curve sketched in figure 1, see e.g. [27]. While ζ_2 is close to 2/3 it has to be a bit larger because experiments show that the slope at zero $\mathrm{d}\zeta_n/\mathrm{d}n$ is larger than 1/3 while $\zeta(3) = 1$ is in agreement with (11). Like in Burgers, the PDF of velocity differences in the inertial interval is not scale invariant in the 3D incompressible turbulence. So far, nobody has been able to find an explicit relation between the anomalous scaling for

3D Navier–Stokes turbulence and either structures or additional integrals of motion. We understand qualitatively the breakdown of scale invariance in Navier–Stokes turbulence and in a related problem of passive scalar turbulence in terms of *statistical Lagrangian integrals of motion* (as opposite to dynamical integrals in the Burgers turbulence), see the following section 2.3. Namely, it is believed that the correlation functions are determined by persistent structures. For example, the second velocity moment must have a scaling (close but not equal to 2/3) of the statistically conserved quantity build out of velocity vectors of two fluid particles and the distance between them: this scaling is determined by the law of de-correlation of two vectors convected by the flow (rather than energy flux constance which determines only the third moment).

2.3. Passive scalar turbulence

Consider a scalar quantity $\theta(\mathbf{r}, t)$ which is passively carried by the fluid flow and is also a subject to molecular diffusion and external source:

$$\partial_t\theta + (\mathbf{v}\cdot\nabla)\theta = \kappa\nabla^2\theta + \varphi. \tag{12}$$

If the source φ produces fluctuations of θ on some scale L then the inhomogeneous velocity field stretches, contracts and folds the field θ producing progressively smaller and smaller scales—this is the mechanism of the scalar cascade. If the rms velocity gradient is Λ then molecular diffusion is substantial at scales less than the diffusion scale $r_d = \sqrt{\kappa/\Lambda}$. For scalar turbulence, the ratio $Pe = L/r_d$, called the Peclet number, plays the role of the Reynolds number. When $Pe \gg 1$, there is an inertial interval with a constant flux of θ^2:

$$\langle(\mathbf{v}_1\cdot\nabla_1 + \mathbf{v}_2\cdot\nabla_2)\theta_1\theta_2\rangle = 2P, \tag{13}$$

where $P = \kappa\langle(\nabla\theta)^2\rangle = \langle\varphi\theta\rangle$ and subscripts denote the spatial points. The correlation function (13) changes sign under the time reflection so its nonzero value means time irreversibility. Finiteness of P at $\kappa \to 0$ signals again a dissipative anomaly.

In considering the passive scalar problem, the velocity statistics is presumed to be given. Still, the correlation function (13) mixes $\mathbf{v}$ and θ and does not generally allow one to make a statement on any correlation function of θ. The proper way to describe the correlation functions of the scalar at scales much larger than the diffusion scale is to employ the Lagrangian description, that is to follow fluid trajectories. Indeed, if we neglect diffusion, then equation (12) can be solved along the characteristics $\mathbf{R}(t)$ which are called Lagrangian trajectories and satisfy $\mathrm{d}\mathbf{R}/\mathrm{d}t = \mathbf{v}(\mathbf{R}, t)$. Presuming zero initial conditions for θ at $t \to -\infty$, we write

$$\theta(\mathbf{R}(t), t) = \int_{-\infty}^{t} \varphi(\mathbf{R}(t'), t')\,\mathrm{d}t'. \tag{14}$$

In that way, the correlation functions of the scalar $F_n = \langle\theta(\mathbf{r}_1, t)\ldots\theta(\mathbf{r}_n, t)\rangle$ can be obtained by integrating the correlation functions of the pumping along the trajectories that satisfy the final conditions $\mathbf{R}_i(t) = \mathbf{r}_i$. We consider a pumping which is Gaussian, statistically homogeneous and isotropic in space and white in time

$$\langle\varphi(\mathbf{r}_1, t_1)\varphi(\mathbf{r}_2, t_2)\rangle = \Phi(|\mathbf{r}_1 - \mathbf{r}_2|)\delta(t_1 - t_2),$$

where the function Φ is constant at $r \ll L$ and goes to zero at $r \gg L$. The pumping provides for symmetry $\theta \to -\theta$ which makes only even correlation functions F_{2n} nonzero. The pair correlation function is as follows:

$$F_2(r, t) = \int_{-\infty}^{t} \Phi(R_{12}(t'))\,\mathrm{d}t'. \tag{15}$$

Here $R_{12}(t') = |\mathbf{R}_1(t') - \mathbf{R}_2(t')|$ is the distance between two trajectories and $R_{12}(t) = r$. The function Φ essentially restricts the integration to the time interval when the distance $R_{12}(t') \leqslant L$. Simply speaking, the stationary pair correlation function of a tracer is $\Phi(0)$ (which is twice the injection rate of θ^2) times the average time $T_2(r, L)$ that two fluid particles spent within the correlation scale of the pumping. The larger the r, the less time it takes for the particles to separate from r to L and the less $F_2(r)$ is. Of course, $T_{12}(r, L)$ depends on the properties of the velocity field. How does one build the Lagrangian description when the velocity is not spatially smooth, for example, that of the energy cascades in the inertial interval with $\delta v \propto r^{1/3}$? Again, the only exact relation one can derive for two fluid particles separated by a distance in the inertial interval is for the Lagrangian time derivative of the squared velocity difference [26]:

$$\left\langle \frac{\mathrm{d}|\delta \mathbf{v}|^2}{\mathrm{d}t} \right\rangle = 2\epsilon$$

—this is the Lagrangian counterpart to (5), (11), (19). One can *assume* that the statistics of the distances between particles is also determined by the energy flux. That assumption leads, in particular, to the Richardson law for the asymptotic growth of the inter-particle distance:

$$\left\langle R_{12}^2(t) \right\rangle \sim \epsilon t^3, \tag{16}$$

first inferred from atmospheric observations (in 1926) and later from experimental data on the energy cascades both in 3D and in 2D. There is no consistent theoretical derivation of (16) and it is unclear whether it is exact (likely to be in 2D) or just approximate (possible in 3D). A semi-heuristic argument usually presented in textbooks is based on the mean-field estimate: $\dot{\mathbf{R}}_{12} = \delta \mathbf{v}(\mathbf{R}_{12}, t) \sim (\epsilon R_{12})^{1/3}$ which upon integration gives $R_{12}^{2/3}(t) - R_{12}^{2/3}(0) \sim \epsilon^{1/3} t$. While this argument is at best a crude estimate in 3D (where there is no definite velocity scaling since every moment has its own exponent ζ_n) we use it to discuss implications for the passive scalar[1].

For two trajectories, the Richardson law gives the separation time from r to L: $T_2(r, L) \sim \epsilon^{-1/3}[L^{2/3} - r^{2/3}]$. Note that $T_2(r, L)$ has a finite limit at $r \to 0$—infinitesimally close trajectories separate in a finite time. This leads to non-uniqueness of Lagrangian trajectories (non-smoothness of the velocity field means that the equation $\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R})$ is non-Lipschitz). As discussed in much detail elsewhere [26, 13], this leads to a finite dissipation of a transported passive scalar even without any molecular diffusion (which corresponds to a dissipative anomaly and time irreversibility). Indeed, substituting $T_2(r, L)$ into (15), one gets the steady-state pair correlation function of the passive scalar: $F_2(r) \sim \Phi(0)\epsilon^{-1/3}[L^{2/3} - r^{2/3}]$ as suggested by Oboukhov (1949) [44] and Corrsin (1952) [21]. The structure function is then $S_2(r) \sim \Phi(0)\epsilon^{-1/3} r^{2/3}$. Experimental measurements of the scaling exponents $\sigma_n = \mathrm{d}\ln S_n(r)/\mathrm{d}\ln r$ generally give σ_2 close to 2/3 but higher exponents deviating from the straight line are even stronger than the exponents of the velocity in 3D as seen in figure 1. Moreover, the scalar exponents σ_n are anomalous even when the advecting velocity has a normal scaling like in the 2D energy cascade (described in section 3.2 below).

To explain the dependence $\sigma(n)$ and describe multi-point correlation functions or high-order structure functions one needs to study multi-particle statistics. Here an important question is what memory of the initial configuration remains when final distances far exceed initial ones. To answer this question one must analyze the conservation laws of turbulent diffusion. We now describe a general concept of conservation laws which, while conserved only on average, still determine the statistical properties of strongly fluctuating systems. In

[1] What matters here and below is that in a non-smooth flow $R_{12}^a(t) - R_{12}^a(0) \sim t$ with $a < 1$, not the precise value of a.

a random system, it is always possible to find some fluctuating quantities which ensemble averages do not change. We now ask a subtler question: is it possible to find quantities that are expected to change on dimensional grounds but stay constant [26, 27]? Let us characterize n fluid particles in a random flow by inter-particle distances R_{ij} (between particles i and j). Consider homogeneous functions f of inter-particle distances with a nonzero degree σ, i.e. $f(\lambda R_{ij}) = \lambda^\sigma f(R_{ij})$. When all the distances grow on average, say according to $\langle R_{ij}^2\rangle \propto t^a$, then one expects that a generic function grows as $f \propto t^{a\sigma/2}$. How are (specific) functions that are conserved on average built, and which σ-s do they have? As the particles move in a random flow, the n-particle cloud grows in size and the fluctuations in the shape of the cloud decrease in magnitude. Therefore, one may look for suitable functions of size and shape that are conserved because the growth of distances is compensated by the decrease of shape fluctuations.

For the simplest case of a Brownian random walk, inter-particle distances grow by the diffusion law: $\langle R_{ij}^2(t)\rangle = R_{ij}^2(0) + \kappa t, \langle R_{ij}^4(t)\rangle = R_{ij}^4(0) + 2(d+2)\left[R_{ij}^2(0)\kappa t + \kappa^2 t^2\right]/d$, etc. Here d is the space dimensionality. Two particles are characterized by a single distance. Any positive power of this distance grows on average. For many particles, one can build conserved quantities by taking the differences where all powers of t cancel out: $f_2 = \langle R_{12}^2 - R_{34}^2\rangle$, $f_4 = \langle 2(d+2)R_{12}^2R_{34}^2 - d\left(R_{12}^4 + R_{34}^4\right)\rangle$, etc. These polynomials are called harmonic since they are zero modes of the Laplacian in the two-dimensional space of $\mathbf{R}_{12}, \mathbf{R}_{13}$. One can write the Laplacian as $\Delta = R^{1-2d}\partial_R R^{2d-1}\partial_R + \Delta_\theta$, where $R^2 = R_{12}^2 + R_{13}^2$ and Δ_θ is the angular Laplacian on $(2d-1)$-dimensional unit sphere. Introducing the angle, $\theta = \arcsin(R_{12}/R)$, which characterizes the shape of the triangle, we see that the conservation of both $f_2 = \langle R^2 \cos 2\theta\rangle$ and $f_4 = \langle R^4[(d+1)\cos^2 2\theta - 1]\rangle$ can also be described as due to cancelation between the growth of the radial part (as powers of t) and the decay of the angular part (as inverse powers of t). For n particles, the polynomial that involves all distances is proportional to R^{2n} (i.e. $\sigma_n = n$) and the respective shape fluctuations decay as t^{-n}.

The scaling exponents of the zero modes are thus determined by the laws that govern decrease of shape fluctuations. The zero modes, which are conserved statistically, exist for turbulent macroscopic diffusion as well. However, there is a major difference since the velocities of different particles are correlated in turbulence. Those mutual correlations make shape fluctuations decay slower than t^{-n} so that the exponents of the zero modes, σ_n, grow with n slower than linearly. This is very much like the total energy of the cloud of attracting particles that does not grow linearly with the number of particles. Indeed, power-law correlations of the velocity field lead to super-diffusive behavior of inter-particle separations: the farther particles are, the faster they tend to move away from each other, as in Richardson's law of diffusion. That is, the system behaves as if there was an attraction between particles that weakens with distance, though, of course, there is no physical interaction among particles (but only mutual correlations because they are inside the correlation radius of the velocity field). While zero modes of multi-particle evolution exist for all velocity fields—from those that are smooth to those that are extremely rough as in Brownian motion—only those non-smooth velocity fields with power-law correlations provide for an anomalous scaling. Zero modes were discovered in [18, 32, 52] and then described in [1, 5, 19].

The existence of multi-particle conservation laws indicates the presence of a long-time memory and is a reflection of coupling among the particles due to the simple fact that they are all in the same velocity field.

Let us now connect these statistical conservation laws (called martingales in probability theory) to an anomalous scaling of fields carried by a turbulent flow. According to (14), the correlation functions of θ are proportional to the times spent by the particles within the correlation scales of the pumping. The structure functions of θ are differences of

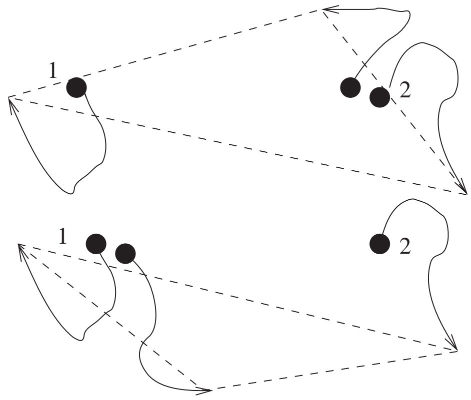

Figure 2. Two configurations (upper and lower) whose difference determines the third structure function.

correlation functions with different initial particle configurations as, for instance, $S_3(r_{12}) \equiv \langle[\theta(\mathbf{r}_1) - \theta(\mathbf{r}_2)]^3\rangle = 3\langle\theta^2(\mathbf{r}_1)\theta(\mathbf{r}_2) - \theta(\mathbf{r}_1)\theta^2(\mathbf{r}_2)\rangle$. In calculating S_3, we are thus comparing two histories: the first one with two particles initially close to the position $\mathbf{r}_1$ and one particle at $\mathbf{r}_2$, and the second one with one particle at $\mathbf{r}_1$ and two particles at $\mathbf{r}_2$— see figure 2. That is, S_3 is proportional to the time during which one can distinguish one history from another, or to the time needed for an elongated triangle to relax to the equilateral shape. This time grows with r_{12} (as it takes longer to forget a more elongated triangle) by the law that can be inferred from the law of the decrease of the shape fluctuations of a triangle.

Quantitative details can be worked out for the white in time velocity called the Kraichnan model [37]. The profound insight of Kraichnan was its spatial rather than temporal non-smoothness of the velocity that is crucial for an anomalous scaling. Analytical and numerical calculations of the scaling exponents σ_n (described in detail in [26]) give σ_n lying on a convex curve (see figure 1) which saturates to a constant at large n according to [1]. Such saturation is a sign that most singular structures in a scalar field are shocks like in Burgers turbulence; the value σ_n at $n \to \infty$ is the fractal codimension of fronts in space [16].

The existence of statistical conserved quantities breaks the scale invariance of scalar statistics in the inertial interval and explains why scalar turbulence knows about pumping ‘more’ than just the value of the flux. Here again the statistics in the inertial interval, apart from the flux of θ^2, depends on the infinity of pumping related parameters. However, these parameters are neither fluxes of θ^n, nor we can interpret them as any other fluxes. At the present level of understanding, we thus describe an anomalous scaling in Burgers and in passive scalar in quite different terms. Of course, the qualitative appeal to structures (shocks) is similar but the nature of the conservation laws is different. The anomalies produced by dynamically conserved quantities (like anomalous scaling in Burgers and time irreversibility in all cases of turbulence) are qualitatively different from the anomalies produced by statistically conserved quantities (like breakdown of scale invariance in passive scalar turbulence). Indeed, dissipation is a singular perturbation which breaks conservation of dynamical integrals of motion and imposes (one or many) flux-constancy conditions, very much similar to quantum anomalies. In contrast, there are no cascades of conserved quantity related to zero modes, nor is their conservation broken by dissipation. Anomalous scaling of zero modes is due to correlations between different fluid trajectories. On the other hand, the two types of anomalies are related intimately: the flux constancy requires a certain degree of velocity non-smoothness, which generally leads to an anomalous scaling of zero modes.

Both symmetries, one broken by pumping (scale invariance) and another by damping (time reversibility) are not restored even when $r/L \to 0$ and $r_d/r \to 0$.

For the vector field (like the velocity or magnetic field in magnetohydrodynamics) the Lagrangian statistical integrals of motion may involve both the coordinate of the fluid particle and the vector it carries. Such integrals of motion were built explicitly and related to the anomalous scaling for the passively advected magnetic field in the Kraichnan ensemble of velocities [26]. Doing that for the velocity that satisfies the 3D Navier–Stokes equation remains a task for the future.

Contrast an anomalous scaling in turbulent direct cascades, which generally involves an infinite number of different exponents (a property also referred to as multifractality [9]), with only a few basic exponents that determine equilibrium critical phenomena.

We thus conclude that in direct cascades we have at least two anomalies:

- finite third moment means time irreversibility even when $\nu \to 0$,
- scale invariance is not restored even when $x/L \to 0$.

3. Inverse cascades

In this chapter, we consider inverse cascades and discover that, while time reversibility remains broken, scale invariance is restored in the inertial interval. Moreover, even wider symmetry of conformal invariance may appear there. A program to extend scale invariance to conformal invariance in turbulence was suggested by Polyakov [47]. Conformal transformations realize non-uniform change of scale (preserving the angles) so that conformal invariance can be thought of as local scale invariance. Specifically, consider turbulence in some connected domain $\mathcal{D} \subset C$ and the family of measures $\mu_{\mathcal{D}}(z_1, \ldots, z_n)$, depending on the points $z_i \in \mathcal{D}$ (for instance, probabilities of the velocity differences in different points). In another domain $\mathcal{D}'$, a random force with the same correlation radius produces turbulence with another family $\mu_{\mathcal{D}'}$. We call the measure conformal invariant if it is invariant with respect to the conformal map $f : \mathcal{D} \to \mathcal{D}'$, that is, $\mu_{\mathcal{D}}(z_1, \ldots, z_n) = \mu_{\mathcal{D}'}(f(z_1), \ldots, f(z_n))$.

3.1. Inverse cascade of a passive scalar in a compressible flow

Similar to (15) one can derive for a passive scalar from (14)

$$\langle \theta(t, \mathbf{r}_1) \ldots \theta(t, \mathbf{r}_{2n}) \rangle = \int_0^t \mathrm{d}t_1 \ldots \mathrm{d}t_n \times \langle \Phi(R(t_1|T, \mathbf{r}_{12})) \ldots \Phi(R(t_n|T, \mathbf{r}_{2n-1,2n})) \rangle + \ldots . \tag{17}$$

The functions Φ in (17) restrict integration to the time intervals where $R_{ij} < L$. If the Lagrangian trajectories separate, the correlation functions reach at long times the stationary form for all r_{ij}. Such steady states correspond to a direct cascade of the tracer (i.e. from large to small scales) considered above. This generally takes place in incompressible and weakly compressible flows.

It is intuitively clear that in compressible flows the regions of compressions can trap fluid particles counteracting their tendency to separate. Indeed, one can show that particles cluster in flows with high enough compressibility [20, 26, 31]. In particular, particles that start from the same point will remain together throughout the evolution. That means that advection preserves all the single-point moments $\langle \theta^N \rangle(t)$. These conservation laws are statistical: the moments are not dynamically conserved in every realization, but their averages over the velocity ensemble are. In the presence of pumping, the moments are the same as for the equation $\partial_t\theta = \varphi$ in the limit $\kappa \to 0$ (nonsingular now). It follows that the single-point statistics is Gaussian, with $\langle\theta^2\rangle = \Phi(0)t$. That growth is produced by the flux of scalar variance toward the large scales—a phenomenon that can be loosely called an inverse cascade of a passive tracer [20, 26, 31]. As is clear from (17), correlation functions at very large scales are related to the probability for initially distant particles to come close (inside the forcing correlation length L). The larger the time the larger the distance starting from which particles come within L. The correlations of the field θ at larger scales are therefore established as time increases, signaling the inverse cascade process.

The uniqueness of the trajectories greatly simplifies the analysis of the PDF $\mathcal{P}(\delta\theta, r)$. Indeed, the structure functions involve initial configurations with just two groups of particles separated by a distance r. The particles explosively separate in the incompressible case and we are immediately back to the full N-particle problem. Conversely, the particles that are initially in the same group remain together if the trajectories are unique. The only relevant degrees of freedom are then given by the intergroup separation and we are reduced to a two-particle dynamics. It is therefore not surprising that the statistics of the passive tracer is scale invariant in the inverse cascade regime [31]. Using the solvable (Kraichnan) model of a short-correlated compressible flow, one can show that multi-point correlation functions of the scalar are scale invariant but not conformal invariant.

An example of a strongly compressible flow is given by Burgers turbulence (1) where there is clustering (in shocks) for the majority of trajectories (full measure in the inviscid limit). Considering passive scalar in such a flow, $\theta_t + u\theta_x - \kappa\Delta\theta = \phi$, we conclude that it undergoes an inverse cascade. The statistics of θ is scale invariant at scales exceeding the correlation scale of the pumping ϕ. While the limit $\kappa \to 0$ is regular (i.e. no dissipative anomaly), the statistics is time irreversible because of the flux toward large scales. It is instructive to compare u and θ which are both Lagrangian invariants (tracers) in the unforced undamped limit. When pumped, the passive quantity θ (and all its powers) go to large scales while all powers of u cascade toward small scales and are absorbed by viscosity. Physically, the difference is evidently due to the fact that the trajectory depends on the value of u it carries, the larger the velocity the faster it ends in a shock and dissipates the energy and other integrals. Formally, for active tracers like u^n one cannot write a formula like (17) obtained by two independent averages over the force and over the trajectories.

3.2. Inverse energy cascades in hydrodynamics

Two-dimensional turbulence is interesting for its own sake and for understanding atmospheric and oceanic turbulence at the scales larger than the atmosphere height and the ocean depth. Taking *curl* of the 2D Navier–Stokes equation one gets

$$d\omega/dt = \partial_t\omega + (\mathbf{v}\cdot\nabla)\omega = \nu\nabla^2\omega. \qquad (18)$$

One sees that the vorticity $\omega = \text{curl}\,\mathbf{v}$ is a Lagrangian invariant like θ but, of course, not passive. The two-dimensional incompressible inviscid flow just transports vorticity from place to place and thus conserves spatial averages of any function of vorticity, $\Omega_n \equiv \int \omega^n\, d\mathbf{r}$. In particular, any inviscid flow conserves the second quadratic invariant (in addition to energy) which is called enstrophy: $\Omega_2 = \int \omega^2\, d\mathbf{r}$. The spectral density of the energy is $|\mathbf{v}_k|^2/2$ while that of the enstrophy is $|\mathbf{k}\times\mathbf{v}_k|^2$. Pumping (at some k_f) generally provides for an input of both E and Ω_2. If there are two inertial intervals (at $k \gg k_f$ and $k \ll k_f$), then there should be two cascades. Indeed, absorbing finite amount of Ω_2 at $k_d \to \infty$ corresponds to an absorption of an infinitely small E. It is thus clear that the flux of E has to go in the opposite direction, that is, to large scales. For the inverse energy cascade in the 2D Navier–Stokes equation, there is no consistent theory except for the flux relation that can be derived similarly to (11):

$$S_3(r) = 4\epsilon r/3. \tag{19}$$

Note that the sign is opposite to (11) due to the opposite direction of the cascade. Again, (19) means that time reversibility is broken in the inverse cascade. Experiments [17, 34, 56] and numerical simulations [8] demonstrate scale-invariant statistics with the vorticity having scaling dimension 2/3: $\omega_r \propto r^{-2/3}$. In particular, $S_2 \propto r^{2/3}$ which corresponds to $E_k \propto k^{-5/3}$. It is ironic that probably the most widely known statement on turbulence, the 5/3 spectrum suggested by Kolmogorov for 3D, is not correct in this case (even though the true scaling is close) while it is probably exact in Kraichnan's inverse 2D cascade. Qualitatively, the absence of anomalous scaling in the inverse cascade seems to be associated with the growth of the typical turnover time (estimated, say, as $r/\sqrt{S_2}$) with the scale. As the inverse cascade proceeds, the fluctuations have enough time to get smoothed out as opposite to the direct cascade in 3D, where the turnover time decreases in the direction of the cascade. Note in passing that passive scalar undergoes direct cascade in the flow of the 2D inverse energy cascade; scalar statistics is not scale invariant since the velocity is non-smooth (compare with the relation between the Lagrangian invariants u and θ for the Burgers turbulence).

The two-dimensional Navier–Stokes equation belongs to a family of models that describe the transport of a scalar quantity by an incompressible velocity related to a scalar by an instantaneous linear scale-invariant relation. Consider a real function of time and coordinates, $a(\mathbf{r}, t)$, which evolves according to the equation

$$\partial a/\partial t + (\mathbf{v} \cdot \nabla)a = f + \nu\Delta a - \alpha a. \tag{20}$$

Here $\mathbf{r} = (x, y)$ belongs to a two-dimensional manifold (plane, disc or torus) where one defines a solenoidal vector field of velocity: $\mathbf{v} = (\partial\Psi/\partial y, -\partial\Psi/\partial x)$. The stream function Ψ is related to the quantity a by a linear scale-invariant relation $\Psi(\mathbf{r}, t) \simeq \int d\mathbf{r}'|\mathbf{r}-\mathbf{r}'|^{m-2}a(\mathbf{r}', t)$. That is, a carried by the velocity $\mathbf{v}$ is pumped by the force f and is dissipated by the viscous and uniform (bottom) friction with the friction coefficients respectively ν and α.

For the models of physical interest, m is integer. The 2D Navier–Stokes equation corresponds to $m = 2$ when the pseudo-scalar $a = \nabla \times \mathbf{v} = \Delta\Psi = \omega$ is the vorticity and $\Psi(\mathbf{r}, t) = -(2\pi)^{-1} \int d\mathbf{r}' \ln|\mathbf{r} - \mathbf{r}'|a(\mathbf{r}', t)$. The case $m = 1$ corresponds to the surface quasi-geostrophic (SQG) model that describes rotating buoyancy-driven flows near a solid surface; a is the temperature in this case [33, 45]. The case $m = -2$ describes large-scale flows of a rotating shallow fluid [40].

The left part of equation (20) conserves scalar variance $\int a^2(r)\, d\mathbf{r}$ and the 'energy' $\int \Psi(r)a(r)\, d\mathbf{r}$, the right-hand side describes generation and dissipation. Two quadratic integrals of motion with a different order of space derivatives mean the existence of two inertial intervals where the nonlinear (inertial) term of (20) dominates and provides for the spectral transfer of the scalar and the energy respectively in the direct and inverse cascades. Here we consider the inverse cascade determined by the 'energy' flux ϵ. Similar to (11), (19) one can derive

$$\langle a_r^3 \rangle \simeq \epsilon r^{2-2m}. \tag{21}$$

The experiments and numerics confirm the relation and show that within the precision determined by the finiteness of the inertial interval and experimental errors, the probability distribution is invariant with respect to global (uniform) scale transformations. The index can be read from (21): $h = (2 - 2m)/3$, in particular $\mathcal{P}(a_r, r) \sim a_r^{-1} f(a_r r^{2/3})$ for $m = 2$ [8, 15, 17, 34, 56] and $\mathcal{P}(a_r, r) \sim a_r^{-1} f(a_r/\ln(k_f r))$ for $m = 1$ [15, 50, 55]. It is this class where the scale invariance was promoted to conformal invariance. Note that the nonlocal relation between the velocity **v** and the field a it carries makes our systems dynamically nonlocal. However, we excite the systems by a noise with the short radius of correlation k_f^{-1} and hope to find locality in statistics. This property takes place for some remarkable class of random curves which we describe now.

3.3. Schramm–Loewner evolution (SLE)

A non-self-intersecting curve growing from the domain boundary can be described by a conformal map of the domain with the curve inside into a domain without the curve. For example, in the simplest case the curve $\gamma(t)$ starts at the real axis of the half-plane H. Here t parameterizes the curve; it should not be confused with the time from (20). The map $g_t{:}H\backslash\gamma(t)\to H$ is fixed by the asymptotics $g_t(z)\sim z+2t/z+\mathrm{O}(1/z^2)$ at infinity. If the curve touches itself, one must define the domain $K(t)$ as the union of the curve and all points that cannot be reached from infinity and consider $g_t : H\backslash K(t)\to H$. The growing tip of the curve is mapped into a real point $\xi(t)$. Loewner found in 1923 that the conformal map $g_t(z)$ and the curve $\gamma(t)$ are fully parameterized by the tip image $\xi(t)$ called the driving function [42]. For that one needs to solve the remarkably simple Loewner equation $\mathrm{d}g_t(z)/\mathrm{d}t = 2[g_t(z)-\xi(t)]^{-1}$. Almost 80 years later, Schramm considered random curves in planar domains and showed (first, in a particular case) that the measure on the curves is conformal invariant if and only if $\xi(t)=\sqrt{\kappa}B_t$, where B_t is a standard one-dimensional Brownian walk [51]. In addition, the measure $\mu_H(\gamma; z_1, z_2)$ on the curves γ connecting z_1 and z_2 is Markovian: if to divide γ into two pieces γ_1 from the boundary z_1 to z, and γ_2 from z to z_2, then the conditional measure is as follows: $\mu_H(\gamma_2|\gamma_1; z_1, z_2) = \mu_{H\backslash\gamma_1}(\gamma_2; z, z_2)$. Diffusivity κ allows one to classify the classes of conformal invariance random curves called SLE_κ. Such curves have been encountered in physics before as the boundaries of clusters of 2D critical phenomena described by conformal field theories. The language and formalism of SLE is a new natural communication tool for physicists and mathematicians, leading to an explosive growth of new results in mathematics, field theory and the theory of critical phenomena [2, 11, 41]. We shall see in the following section that SLE is encountered in hydrodynamics as well.

Let us list here few basic facts about SLE curves. When $\kappa = 0$, γ is a vertical straight line. The larger the κ, the more the curve wiggles. The curve is simple (i.e. with probability 1 does not touch itself or the real axis) when $0 \leqslant \kappa < 4$. For SLE_κ with $4 \leqslant \kappa < 8$, the curve touches itself but does not fill the space. In this case, one can define an external perimeter (as a part one can reach from infinity) which belongs to a dual class SLE_{κ_*} with $\kappa_* = 16/\kappa$ [3, 22, 49]. The fractal dimension of SLE_κ curves is $D_\kappa = 1+\kappa/8$ for $\kappa < 8$.

Among the dual pairs, κ and κ_*, one is special from the viewpoint of locality. The curves from SLE_6 do not feel the boundary until they touch it (a rigorous definition of that property called SLE locality can be found in [41]). The dual curve $\mathrm{SLE}_{8/3}$ has the 'restriction property': the statistics of the curves conditioned not to visit some region is the same as in the domain without this region. Intuitively, one can appreciate these properties by considering lattice (discrete) models which turn into the respective SLE in the continuous limit [2, 41]. For example, consider a honeycomb lattice. A random walk along the bonds starts from the boundary point that has all black hexagons to the left and white to the right and keeps that property as it moves, turning right/left as it meets black/white hexagons. SLE_6 is obtained from the classical model of critical percolation when hexagons get their colors independently with the probability 1/2. $\mathrm{SLE}_{8/3}$ corresponds to a self-avoiding random walk when every bond is visited only once. Also the value $\kappa = 4$ is special because it is self-dual corresponding to the so-called harmonic navigator. In this case, the probability of the color for the hexagon encountered is determined by the harmonic function defined in the domain with the boundary that includes the hexagons colored before; in other words, a new random walk starts from the hexagon and colors it by the color of the boundary the walk hits [2, 41, 52]. Both SLE_6 and SLE_4 appear as isolines of Gaussian random fields. If one considers the surface of a random function of two variables, $a(x, y)$, as a landscape during a great flood then at some water level the probability to sail across is equal to the probability to walk. At this level, the shoreline belongs to SLE_6 (critical percolation) if the correlation functions of $a(x, y)$ decay sufficiently fast. In particular, a non-rigorous but plausible Harris criterion claims that if $\langle a(\mathbf{r})a(0)\rangle \sim r^{-2h}$ and $h \geqslant 3/4$, then isolines of the Gaussian field a are equivalent to critical percolation [57]. This follows from the fact that when a is non-zero, percolation is non-critical even for a short-correlated field, and a finite correlation length appears which scales as $l_c \propto a^{-4/3}$; this means that the non-zero isoline cannot be distinguished from the zero isoline at scales shorter than l_c. In other words, on a scale r one is allowed fluctuation of the field less than $r^{-4/3}$. Therefore, if on the scale r the fluctuations are of the size r^{-h} with $h \geqslant 3/4$ then the fluctuations of the field $a(x, y)$ are small and its nodal line belongs to SLE_6. In contrast, isolines of the Gaussian field a with $h < 3/4$ are not equivalent to critical percolation, i.e. do not belong to SLE_6. As far as SLE_4 is concerned, this class contains isolines of Gaussian (free) fields with $\langle a(\mathbf{r})a(0)\rangle \sim \ln r$ [12, 14, 52]. How is all that related to turbulence where the only thing we are sure about is it being non-Gaussian (because the flux makes the third moment nonzero)?

3.4. Isolines in turbulence

The fractal dimension of SLE_κ curves is known to be $D_\kappa = 1 + \kappa/8$ for $\kappa < 8$. To establish a possible link between turbulence and critical phenomena, let us try to relate Richardson phenomenology to the fractal dimension of the nodal lines. Note that we ought to distinguish between the dimensionality 2 of the full zero-level set (which is space-filling) and a single nodal line that encloses a large-scale cluster. Consider the cluster of gyration radius L which has the 'outer boundary' of perimeter P (that boundary is the part of the nodal line accessible from the outside). The velocity difference $\delta v(l)$ grows with the distance l. This means that the two point distance, which satisfies $\mathrm{d}l/\mathrm{d}t = \delta v(l)$ cannot grow more slowly than t. In our case, velocity scaling $v_l \sim l^{(m-1)/3}$ tells us that for $m > 1$ we have the velocity difference $\delta v(l) \propto l^{(m-1)/3}$ and the Richardson law $l(t) \sim t^{3/(4-m)}$, while for $m < 1$ we have $\delta v(l) \simeq v_{\mathrm{rms}}$ and $l(t) \propto t$. Perimeter P and gyration radius L can be related by *assuming* that their ratio, P/L, which is proportional to the number of folds, grows as a random walk, i.e. $t^{1/2}$. The gyration radius grows as two-point distance $L(t) \propto l(t)$ which gives $P \propto Lt^{1/2} \propto L^{(10-m)/6}$ for $m \geqslant 1$ and $P \propto L^{3/2}$ for $0 \leqslant m \leqslant 1$. Of course, contours reconnect and disconnect but the scaling must hold for every part and for the whole product of reconnections. Therefore, we expect the fractal dimension of the external perimeter of the nodal line to be

$$D = \begin{cases} (10-m)/6 & \text{for} \quad m \geqslant 1, \\ 3/2 & \text{for} \quad 0 \leqslant m \leqslant 1. \end{cases} \tag{22}$$

In particular, for the Navier–Stokes equation $m = 2$ and $P \propto L^{4/3}$, i.e. the fractal dimension of the exterior of the vorticity cluster is expected to be 4/3. This remarkable dimension corresponds to a self-avoiding random walk (SLE curve) which is also known to be an exterior boundary (without self-intersections) of the percolation cluster (yet another SLE curve). Figure 3 shows a nodal line of vorticity obtained by a numerical solution of (20) with $m = 2$ on a torus (that is, the 2D Navier–Stokes equation with periodic boundary conditions and added external force and uniform friction); the details can be found in [6, 8]. The force scale is $l_f = 2\pi/k_f = 0.05$. The curve looks fractal at scales exceeding l_f, i.e. in

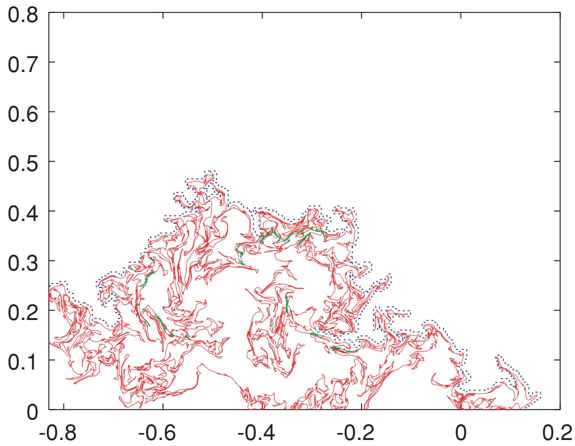

Figure 3. A portion of a candidate SLE trace obtained from the vorticity field. The figure has been adapted from [6].

the interval of an inverse cascade. Indeed, the length P grows nonlinearly with the end-to-end distance L [6]. Power-law exponents of this growth for the curve and its external perimeter are found to be close within the resolution to the dimensionalities 7/4, 4/3 of the dual pair $SLE_6/SLE_{8/3}$ (historically, dimensionality 4/3 of the external perimeter has been guessed from the Kolmogorov–Kraichnan scaling $a_r \sim r^{-2/3}$, which stimulated the search for SLE in turbulence [6]). Let us briefly describe how we identified possible curves from an SLE class and determined the driving function $\xi(t)$. We drew quite arbitrarily a straight line to be a real axis and at the end checked that translations and rotations of the axis did not change the results. We then start from the intersection of a zero isoline and the axis and move along the curve or along the axis (when we return to it) preserving orientation, i.e. keeping positive vorticity always to the right. Such a procedure faithfully reproduces the statistics only in the local case, indeed we expected (and found!) $\kappa \approx 6$. We then divided our curve into small straight segments and approximated the family of conformal maps $g_t(z)$ by a discrete set of standard conformal maps absorbing one segment one by one [6]. The resulting set of 'times' t_i and values ξ_i defines the driving function $\xi(t)$. The only thing left is to run the Schramm test, i.e. to check how well this function corresponds to a Brownian walk. The data presented by the upward oriented triangles in figure 4 show that the ensemble average $\langle\xi(t)^2\rangle$ indeed grows linearly in time: the diffusion coefficient κ is very close to the value 6, with an accuracy of 5% (lower inset). The probability distribution functions of $\xi(t)/\sqrt{\kappa t}$ collapse onto a standard Gaussian distribution at all times t (upper inset). Therefore, we expect that in the limit of vanishingly small L_f the driving $\xi(t)$ tends to true Brownian motion and zero-vorticity lines become SLE_κ traces with κ very close to 6. Note that the vorticity field has $h = 2/3 < 3/4$, that is the Harris criterion is violated. However, our field is non-Gaussian—while the probability distribution looks Gaussian, the deviations are measurable including the third moment [8, 15, 17, 34, 56]. Triangles pointing down on the lower inset are obtained for the isolines of a Gaussian field having the same Fourier spectrum as vorticity but randomized phases. Apparently, our accuracy is sufficient to make sure that it does not correspond to any SLE including SLE_6. Indeed, $E[\xi^2]/t \equiv \langle\xi^2\rangle/t$ is not constant and approaches the limiting value $\kappa = 6$ only at scales exceeding $2\pi/k_\alpha$ where the power-law correlation is already cut-off by friction and the field becomes truly uncorrelated. Something remarkable happens here:

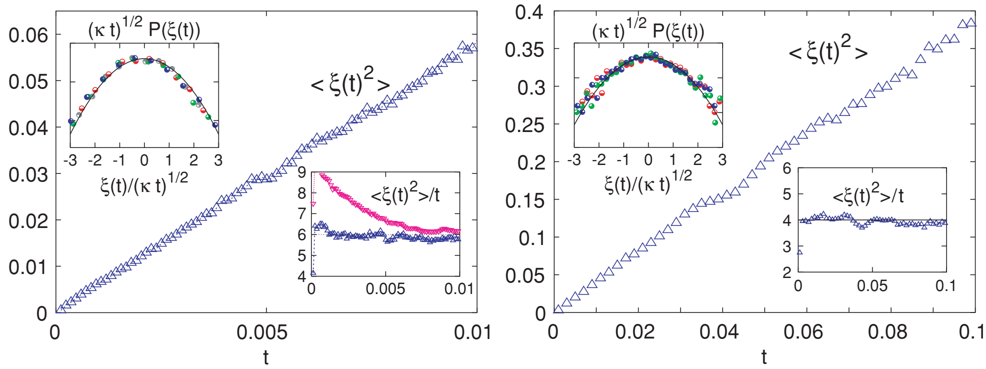

Figure 4. Demonstration of conformal invariance of the isolines of vorticity in the Euler equation (left) and of the temperature in the surface geostrophic model (right). The driving function is an effective diffusion process with diffusion coefficient $\kappa = 6 \pm 0.3$ (left, [6]) and with $\kappa = 4 \pm 0.2$ (right, [7]). Right (lower) inset: triangles pointing up correspond to the vorticity, triangles pointing down to the Gaussian field with the same second moment. Left (upper) insets: the probability density function of the re-scaled driving function $\xi(t)/\sqrt{\kappa t}$ at four different times $t = 0.0012, 0.003, 0.006, 0.009$ (left) and $t = 0.02, 0.04, 0.08$ (right); the solid lines are the Gaussian distribution $g(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

non-Gaussianity of the vorticity field, i.e. multi-point correlations, somehow conspire to make the zero-vorticity line statistically equivalent to the isoline of a short-correlated field even though the pair correlation function decays slower than the Harris criterion requires.

The identification of isolines as SLE$_\kappa$ curves allows us to apply powerful techniques borrowed from the theory of stochastic differential equations and conformal mapping theory and to obtain analytic predictions for some nontrivial statistical properties of lines, vortices and clusters in turbulence. For example, vorticity nodal lines are boundaries of vorticity clusters. The statements from the percolation theory that the probability of a cluster (island) decays with area as $s^{-96/91}$, and with the perimeter as $P^{-8/7}$, can be directly confirmed for turbulence [6]. Moreover, SLE allows exact analytic formulae for the probabilities that a nodal line crosses different figures (triangles, rectangles etc). Such probabilities are determined by the second-order ordinary differential equation and are expressed via hypergeometric functions, which miraculously describe turbulence data. It is worth stressing that the maximum one aspired to in turbulence theory before was to predict a single number (usually a scaling exponent and often from dimensional reasoning); now we are able to predict non-trivial functions. An ability to make exact predictions rather than order-of-magnitude estimates is heartening too. Most important though is that the correspondence between the nodal lines in turbulence and SLE hints at some fundamental properties of hydrodynamic equations which we do not yet grasp.

Let us now describe briefly the results for the surface quasi-geostrophic model, $m = 1$. In this case, the zero-temperature isoline crosses the straight line rarely, so that we simply choose as the candidates for SLE the pieces of the curve returning to the line at the distance far exceeding $2\pi/k_\alpha$. The algorithm to extract the driving function is described in [7]. The procedure was checked first by applying it to a self-avoiding random walk where it gave the right value $\kappa = 8/3$ with an accuracy better than 5%. Applying the procedure to the surface quasi-geostrophic model we obtain $\xi(t)$, whose statistics converges at $l_f^2 < \kappa t < 2\pi/k_\alpha$ to Gaussian statistics with $\langle \xi^2(t) \rangle = \kappa t$ and $\kappa = 4 \pm 0.2$, i.e. within the temperature isolines behave locally as curves from SLE_4. Worth stressing is the fact that the temperature field is substantially non-Gaussian [7, 15] so that it is completely unclear how it can have isolines with the same statistics as that of the isolines of the free Gaussian field with the same second moment.

It is remarked that if the contour $z(l)$ belongs to the class SLE_κ, then the unit vector z_l has a Gaussian statistics with the second moment proportional to the logarithm of the contour length. This property has also been found for the isolines of temperature and vorticity for both models [6, 7].

Musacchio studied numerically the family of models for different m and established that for all cases the SLE property of lines took place [43]. The connection between κ, D and α is non-trivial: κ, D change linearly with m for $1 \leqslant m \leqslant 2$ and freeze at $\kappa = 4, D = 3/2$ for $0 \leqslant m \leqslant 1$ in agreement with (22) while the scaling exponent h changes according to $h = (2 - 2m)/3$ for the whole interval $0 \leqslant m \leqslant 2$. For negative m, all this is true for Ψ instead of a [43].

4. Conclusion

Turbulence statistics is always time irreversible. Generally, it seems natural that the statistics within the pumping correlation scale (direct cascade) is more sensitive to the details of the pumping statistics than the statistics at much larger scales (inverse cascade). For direct cascades, the symmetries broken by pumping (scale invariance, isotropy) generally are not restored in the inertial interval. In other words, the statistics at however small scales is sensitive to the characteristics of pumping besides the flux. That can be alternatively explained in terms of structures or in terms of conservation laws, either dynamical or statistical (zero modes). Inverse cascades in systems with strong interaction may be not only scale invariant but also conformal invariant. It is an example of emerging symmetry.

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